Gravity Tunnel Explanation

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Oslo - 60° N 11° E
Honolulu - 21° N 158° W

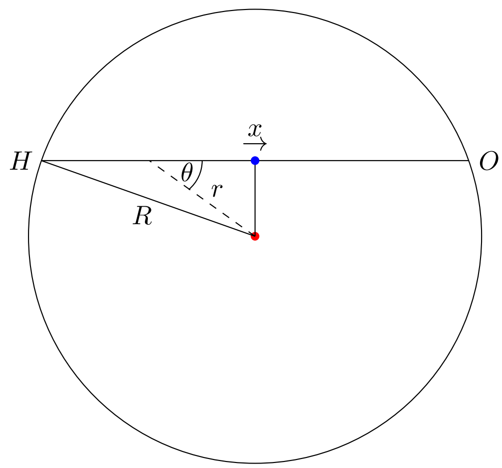

Figure 1: Gravity Tunnel

1 Calculating time taken to fall from Oslo to Honolulu

In Newtonian Gravity, only the mass below the point r attracts it. So the mass below x can be described as

$$\text{Mass below x, } M_r = M\frac{r^3}{R^3} \tag{1}$$

Then using Newton's law of universal gravitation, the centripetal force at a distance r from the center of the Earth can be described as

$$F_{centripetal} = \frac{GM_r m}{r^2} \tag{2}$$

The force acting in the line OH can be written as $F_{centripetal} \times \cos\theta$. This can then be simplified to give the equation

$$F_{OH} = -\frac{Fx}{r} \tag{3}$$

$$F_{OH} = -\frac{GM_r mx}{R^3} \tag{4}$$

Assuming the only force acting on the falling object is the centripetal force, this force can also be described as

$$F_{OH} = m\ddot{x} \tag{5}$$

Equating the two equations

$$m\ddot{x} = -\frac{GM_r mx}{R^3} \tag{6}$$

$$\ddot{x} + \frac{GM_r x}{R^3} = 0 \tag{7}$$

Notice that this is a Simple Harmonic Motion equation. This gives ω as $\sqrt{\frac{GM_r}{R^3}}$. Solving the differential equation

$$x = Acos(\omega t) \tag{8}$$

Therefore the time period of this can then be calculated

$$\omega = \frac{2\pi}{T} \tag{9}$$

$$\sqrt{\frac{GM_r}{R^3}} = \frac{2\pi}{T} \tag{10}$$

$$T = 2\pi\sqrt{\frac{R^3}{GM_r}} \tag{11}$$

Using $R = 6.4 \times 10^3$ km, $G = 6.7 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, $M = 6 \times 10^{24}$ kg and $\pi = 3.14$

$$T \approx 5071 \text{ s} \tag{12}$$

This is the time taken for a complete oscillation, from O to H and then back to O. So the time taken to fall from Oslo to Honolulu is half of this time

$$\text{Time taken to fall from Oslo to Honolulu} \approx 2536 \text{ s} \tag{13}$$

2 Length of the tunnel, maximum acceleration and velocity

We are given the geodetic distance between Oslo and Honolulu as $S = 10920$ km. The angle between OH is then found

$$S = \theta r \tag{14}$$
$$10.920 \times 10^6 = \theta \times 6.4 \times 10^6 \tag{15}$$
$$\theta \approx 1.7^c \tag{16}$$

Then the length of the tunnel is the chord between O and H, Q

$$Q = 2R\frac{\sin\theta}{2} \tag{17}$$
$$Q \approx 9643 \text{ km} \tag{18}$$

The displacement from the middle of the tunnel is described by equation 8. This can be differentiated to give velocity

$$v = -A\omega \sin \omega t \tag{19}$$

The amplitude A, is half of the length of the tunnel. So $A \approx 4822$ km. The velocity cannot be greater than $-A\omega$ as sin cannot be greater than 1. So the maximum velocity is

$$v_{max} = -A\omega \tag{20}$$
$$v_{max} \approx 5970 \text{ ms}^{-1} \tag{21}$$

The velocity equation can then be differentiated again to give acceleration.

$$a = -A\omega^2 \cos \omega t \tag{22}$$

The acceleration cannot be greater than $-A\omega^2$ as cos cannot be greater than 1. So maximum acceleration is

$$a_{max} = -A\omega^2 \tag{23}$$
$$a_{max} \approx 7.4 \text{ ms}^{-2} \tag{24}$$